Finiteness of the total first curvature of a non-closed curve in $\mathbb{E}^n$

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Abstract

We consider a regular smooth curve in $\mathbb{E}^n$ such that its coordinates' components are the fundamental solutions of the differential equation $y^{(n)}(x)-y(x)=0$, $x \in \mathbb{R}$ of order n. We show that the total first curvature of this curve is infinite for odd n and is finite for even n.

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1 Introduction

The study of finiteness of the total curvature of curve has been examined by W. Fenchel, I. Fary, J. W. Milnor and others. In recent years, the total curvature of curve has been discussed in [1, 6, 7, 8, 9, 10]. These discussions are based on closed curves, polygonal curves, knotted curves, curves with fixed endpoints, curves with finite length or pursuit curves. Our discussion is based on non-closed smooth curves with infinite length. We study the real fundamental solutions of a differential equation of order $n(\geq 2)$: $y^{(n)}(x) - y(x) = 0$, $x \in \mathbb{R}$, are given by

(*i*) If $n = 2$, then $e^x,\ e^{-x}$.

(*ii*) If $n = 2m + 1$, then
$$e^{\alpha_1 x}\cos(\beta_1 x),\ e^{\alpha_1 x}\sin(\beta_1 x),\ \cdots,\ e^{\alpha_m x}\cos(\beta_m x),\ e^{\alpha_m x}\sin(\beta_m x),\ e^x.$$

(*iii*) If $n = 2m + 2$, then
$$e^{\alpha_1 x}\cos(\beta_1 x),\ e^{\alpha_1 x}\sin(\beta_1 x),\ \cdots,\ e^{\alpha_m x}\cos(\beta_m x),\ e^{\alpha_m x}\sin(\beta_m x),\ e^x,\ e^{-x}.$$

Here, complex numbers $\lambda_k = \alpha_k \pm \beta_k\sqrt{-1}$ $(k = 1, 2, \cdots, m)$ are solutions (without 1 and -1) of the characteristic polynomial equation $P(\lambda) = \lambda^n - 1 = 0$ of the differential equation $y^{(n)}(x) - y(x) = 0$, $x \in \mathbb{R}$ [4]. Then we define a regular smooth curve $C_n \mid_{-\infty}^{+\infty}$ in $\mathbb{E}^n$ such that its coordinates' components are the above fundamental solutions. Here the parameter t of the curve $C_n \mid_{-\infty}^{+\infty}$ is not an arc-length parameter in general and $\mid_{-\infty}^{+\infty}$

denotes the range $(-\infty, +\infty)$ of parameter t. We take "smooth" to mean "of class C^∞". We consider curves $C_n\mid_{-\infty}^{0}$ and $C_n\mid_{0}^{+\infty}$ that are sub-arcs of $C_n\mid_{-\infty}^{+\infty}$. These curves $C_n\mid_{-\infty}^{0}$, $C_n\mid_{0}^{+\infty}$ and $C_n\mid_{-\infty}^{+\infty}$ in $\mathbb{E}^n$ are of non-closed.

In the present paper, we calculate the total first curvature [2] of the curve $C_n\mid_{-\infty}^{+\infty}$ in $\mathbb{E}^n$. Our result is the following:

Main Theorem *(1) The curves $C_n\mid_{-\infty}^{0}$, $C_n\mid_{0}^{+\infty}$ and $C_n\mid_{-\infty}^{+\infty}$ are of infinite length.*
(2) For an odd number n, the curve $C_n\mid_{-\infty}^{0}$ is of infinite total first curvature, and $C_n\mid_{0}^{+\infty}$ is of finite total first curvature, that is, the curve $C_n\mid_{-\infty}^{+\infty}$ is of infinite total first curvature.
(3) For an even number n, the curve $C_n\mid_{-\infty}^{+\infty}$ is of finite total first curvature.

2 Definition of a curve $C_n\mid_{-\infty}^{+\infty}$ in $\mathbb{E}^n$

We denote $\mathbb{E}^n$ the Euclidean n-space. Let $C_n\mid_{-\infty}^{+\infty}$ be a regular smooth curve in $\mathbb{E}^n$ given by a mapping

$$\mathbf{x} : (-\infty, +\infty) \ni t \longmapsto \mathbf{x}(t) \in \mathbb{E}^n,$$

where $\mathbf{x}(t)$ is defined by

(i) In the case of $n = 2$,

$$\mathbf{x}(t) = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}, \quad t \in (-\infty, +\infty).$$

(ii) In the case of $n = 2m + 1$,

$$\mathbf{x}(t) = \begin{bmatrix} e^{\alpha_1 t}\cos(\beta_1 t) \\ e^{\alpha_1 t}\sin(\beta_1 t) \\ \cdots \\ e^{\alpha_m t}\cos(\beta_m t) \\ e^{\alpha_m t}\sin(\beta_m t) \\ e^t \end{bmatrix}, \quad t \in (-\infty, +\infty).$$

(iii) In the case of $n = 2m + 2$,

$$\mathbf{x}(t) = \begin{bmatrix} e^{\alpha_1 t}\cos(\beta_1 t) \\ e^{\alpha_1 t}\sin(\beta_1 t) \\ \cdots \\ e^{\alpha_m t}\cos(\beta_m t) \\ e^{\alpha_m t}\sin(\beta_m t) \\ e^{t} \\ e^{-t} \end{bmatrix}, \quad t \in (-\infty, +\infty).$$

Here, complex numbers $\lambda_k = \alpha_k \pm \beta_k\sqrt{-1}$ $(k = 1, 2, \cdots, m)$ are solutions (without 1 and -1) of the polynomial equation $P(\lambda) = \lambda^n - 1 = 0$. In this paper, we assume that

$$-1 < \alpha_m < \alpha_{m-1} < \cdots < \alpha_2 < \alpha_1 < 1.$$

Then we have easily the following:

Proposition 1 *The curve $C_n \,|_{-\infty}^{+\infty}$ in $\mathbb{E}^n$ is non-closed and satisfies differential equations $\mathbf{x}^{(n)}(t) = \mathbf{x}(t)$ and $\mathbf{x}^{(q)}(t) \neq \mathbf{x}(t)$, for any $t \in \mathbb{R}$ and $q = 1, 2, \cdots, n-1$.*

Proof. For an integer p, we define constants A_p and B_p by

$$\frac{\mathrm{d}^p e^{\alpha_k t}\cos(\beta_k t)}{\mathrm{d}t^p} = A_p e^{\alpha_k t}\cos(\beta_k t) + B_p e^{\alpha_k t}\sin(\beta_k t).$$

Then we have

$$A_1 = \alpha_k, \quad B_1 = -\beta_k,$$
$$A_{p+1} = \alpha_k A_p + \beta_k B_p, \quad B_{p+1} = -\beta_k A_p + \alpha_k B_p.$$

On the other hand, we define real constants a_p and b_p by

$$(\alpha_k - \beta_k\sqrt{-1})^p = a_p + b_p\sqrt{-1}.$$

Then we have

$$a_1 = \alpha_k, \quad b_1 = -\beta_k,$$
$$a_{p+1} = \alpha_k a_p + \beta_k b_p, \quad b_{p+1} = -\beta_k a_p + \alpha_k b_p.$$

Thus we have $A_p = a_p$ and $B_p = b_p$ for each p. Since it holds that $(\alpha_k - \beta_k\sqrt{-1})^n = 1$, we have $a_n = 1$ and $b_n = 0$ so that $A_n = 1$ and $B_n = 0$. Therefore, we have

$$\frac{\mathrm{d}^n e^{\alpha_k t}\cos(\beta_k t)}{\mathrm{d}t^n} = e^{\alpha_k t}\cos(\beta_k t)$$

and

$$\frac{\mathrm{d}^q e^{\alpha_k t}\cos(\beta_k t)}{\mathrm{d}t^q} \neq e^{\alpha_k t}\cos(\beta_k t)$$

for $q = 1, 2, \cdots, n-1$. Next, for an integer p, we define constants C_p and D_p by

$$\frac{\mathrm{d}^p e^{\alpha_k t}\sin(\beta_k t)}{\mathrm{d}t^p} = C_p e^{\alpha_k t}\sin(\beta_k t) + D_p e^{\alpha_k t}\cos(\beta_k t).$$

Then we have

$$C_1 = \alpha_k, \quad D_1 = \beta_k,$$
$$C_{p+1} = \alpha_k C_p - \beta_k D_p, \quad D_{p+1} = \beta_k C_p + \alpha_k D_p.$$

On the other hand, we define real constants c_p and d_p by

$$(\alpha_k + \beta_k\sqrt{-1})^p = c_p + d_p\sqrt{-1}.$$

Then we have

$$c_1 = \alpha_k, \quad d_1 = \beta_k,$$
$$c_{p+1} = \alpha_k c_p - \beta_k d_p, \quad d_{p+1} = \beta_k c_p + \alpha_k d_p.$$

Thus we have $C_p = c_p$ and $D_p = d_p$ for each p. Since it holds that $(\alpha_k + \beta_k\sqrt{-1})^n = 1$, we have $c_n = 1$ and $d_n = 0$ so that $C_n = 1$ and $D_n = 0$. Thus we have

$$\frac{\mathrm{d}^n e^{\alpha_k t}\sin(\beta_k t)}{\mathrm{d}t^n} = e^{\alpha_k t}\sin(\beta_k t)$$

and

$$\frac{\mathrm{d}^q e^{\alpha_k t}\sin(\beta_k t)}{\mathrm{d}t^q} \neq e^{\alpha_k t}\sin(\beta_k t)$$

for $q = 1, 2, \cdots, n-1$. Therefore, we have $\mathbf{x}^{(n)}(t) = \mathbf{x}(t)$ and $\mathbf{x}^{(q)}(t) \neq \mathbf{x}(t)$, for any $t \in \mathbb{R}$ and $q = 1, 2, \cdots, n-1$. This complete the proof.

Proposition 2 *(1)* $(\alpha_k)^2 + (\beta_k)^2 = 1$ *for* $k = 1, 2, \cdots, m$.

(2) In the case of $n = 2m+1$*:* $-1 < \alpha_m \leq -\frac{1}{2}$ *and* $\mid \alpha_m \mid = \max_{k=1,\cdots,m}\{\mid \alpha_k \mid\}$.

(3) In the case of $n = 2m+2$*:*

(i) if $m = 2p+1$ *(that is, m is an odd number), then*

$$\begin{aligned}\alpha_{2p+1} &= -\alpha_1\\ \alpha_{2p} &= -\alpha_2\\ &\cdots\\ \alpha_{p+2} &= -\alpha_p\\ \alpha_{p+1} &= 0,\end{aligned}$$

(ii) if $m = 2p$ (that is, m is an even number), then

$$\alpha_{2p} = -\alpha_1$$
$$\alpha_{2p-1} = -\alpha_2$$
$$\cdots$$
$$\alpha_{p+1} = -\alpha_p.$$

Let $C_n \mid_{-\infty}^{0}$ and $C_n \mid_{0}^{+\infty}$ be sub-arcs of $C_n \mid_{-\infty}^{+\infty}$, that is, the curves $C_n \mid_{-\infty}^{0}$ and $C_n \mid_{0}^{+\infty}$ are given by $\mathbf{x} : (-\infty, 0] \ni t \longmapsto \mathbf{x}(t) \in \mathbb{E}^n$ and $\mathbf{x} : [0, +\infty) \ni t \longmapsto \mathbf{x}(t) \in \mathbb{E}^n$, respectively.

3 First curvature function

Let $<,>$ and $\| \ \|$ be the canonical inner product and the canonical norm in $\mathbb{E}^n$, respectively. If $n \geq 3$, then the first curvature function k_1 of the curve $C_n \mid_{-\infty}^{+\infty}$ in $\mathbb{E}^n$ is given by

$$k_1(t) = \frac{\| \dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t) \|}{\| \dot{\mathbf{x}}(t) \|^3}$$

for any $t \in (-\infty, +\infty)$, where $\dot{\mathbf{x}}(t) = \dfrac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}$ and $\ddot{\mathbf{x}}(t) = \dfrac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2}$, and

$$\| \dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t) \|^2 = \det \begin{bmatrix} < \dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t) > & < \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t) > \\ < \ddot{\mathbf{x}}(t), \dot{\mathbf{x}}(t) > & < \ddot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t) > \end{bmatrix}$$

for any $t \in (-\infty, +\infty)$ [3]. If $n = 2$, then k_1 is given by

$$k_1(t) = \frac{\det [\dot{\mathbf{x}}(t) \, \ddot{\mathbf{x}}(t)]}{\| \dot{\mathbf{x}}(t) \|^3}$$

for any $t \in (-\infty, +\infty)$ [3]. Now, by Proposition 2 (1), we show the concrete forms of $k_1(t)$ for (i) $n = 2$, (ii) $n = 2m + 1$, (iii) $n = 2m + 2$ as follows:

(i) $n = 2$:

$$\| \dot{\mathbf{x}}(t) \|^2 = e^{2t} + e^{-2t}, \quad \| \ddot{\mathbf{x}}(t) \|^2 = e^{2t} + e^{-2t},$$
$$\det [\dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)] = 2.$$

Thus we have

$$k_1(t) = \frac{2}{\left(\sqrt{e^{2t} + e^{-2t}}\right)^3}$$

for any $t \in (-\infty, +\infty)$.

(*ii*) $n = 2m+1$:

$$\| \dot{\mathbf{x}}(t) \|^2 = \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) + e^{2t}, \quad \| \ddot{\mathbf{x}}(t) \|^2 = \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) + e^{2t},$$
$$< \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t) >= \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right) + e^{2t}$$

and

$$\begin{aligned} &\| \dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t) \|^2 \\ &= \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right)^2 + 2e^{2t} \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) - 2e^{2t} \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right) \end{aligned}$$

for any $t \in (-\infty, +\infty)$. Thus we have

$$\begin{aligned} k_1(t) = &\left[\left(\sum_{k=1}^{m} e^{2\alpha_k t} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right)^2 \right. \\ &\left. + 2e^{2t} \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) - 2e^{2t} \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right) \right]^{1/2} \times \left[\left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) + e^{2t} \right]^{-3/2} \end{aligned}$$

for any $t \in (-\infty, +\infty)$.

(*iii*) $n = 2m+2$:

$$\| \dot{\mathbf{x}}(t) \|^2 = \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) + e^{2t} + e^{-2t}, \quad \| \ddot{\mathbf{x}}(t) \|^2 = \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) + e^{2t} + e^{-2t},$$
$$< \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t) >= \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right) + e^{2t} - e^{-2t}$$

and

$$\begin{aligned} &\| \dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t) \|^2 \\ &= \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right)^2 + 2e^{2t} \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) - 2e^{2t} \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right) \\ &\quad + 2e^{-2t} \left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) + 2e^{-2t} \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right) + 4 \end{aligned}$$

for any $t \in (-\infty, +\infty)$. Thus we have

$$
\begin{aligned}
k_1(t) = &\left[\left(\sum_{k=1}^{m} e^{2\alpha_k t} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right)^2 + 2e^{2t} \left(\left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) - \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right) \right) \right. \\
&\left. + 2e^{-2t} \left(\left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) + \left(\sum_{k=1}^{m} \alpha_k e^{2\alpha_k t} \right) \right) + 4 \right]^{1/2} \\
&\times \left[\left(\sum_{k=1}^{m} e^{2\alpha_k t} \right) + e^{2t} + e^{-2t} \right]^{-3/2}
\end{aligned}
$$

for any $t \in (-\infty, +\infty)$.

By Proposition 2 (3), we have

Proposition 3 *If n is even, then it holds $\| \dot{\mathbf{x}}(-t) \| = \| \dot{\mathbf{x}}(t) \|$ and $k_1(-t) = k_1(t)$ for any $t \in (-\infty, +\infty)$.*

For the length of the curve $C_n \mid_{-\infty}^{+\infty}$, we have

Proposition 4 *The curve $C_n \mid_{-\infty}^{+\infty}$ in $\mathbb{E}^n$ is of infinite length.*

Proof. (1)(i) Case of $C_n \mid_{-\infty}^{0}$ and an odd n:
Since we have $\| \dot{\mathbf{x}}(t) \| > e^{\alpha_m t}$ and $-1 < \alpha_m \leq -\frac{1}{2}$, we have, for a large positive number a,

$$
\begin{aligned}
\int_{-\infty}^{0} \| \dot{\mathbf{x}}(t) \| \, \mathrm{d}t &= \lim_{a \to -\infty} \left(\int_{a}^{0} \| \dot{\mathbf{x}}(t) \| \, \mathrm{d}t \right) \\
&> \lim_{a \to -\infty} \left(\int_{a}^{0} e^{\alpha_m t} \, \mathrm{d}t \right) \\
&= \lim_{a \to -\infty} \left\{ (\alpha_m)^{-1} \left(1 - e^{(\alpha_m) a} \right) \right\} \\
&= \lim_{a \to -\infty} \left\{ -(\mid \alpha_m \mid)^{-1} \left(1 - e^{-|\alpha_m| a} \right) \right\} \\
&= +\infty.
\end{aligned}
$$

(1)(ii) Case of $C_n \mid_{-\infty}^{0}$ and an even n:
Since we have $\parallel \dot{\mathbf{x}}(t) \parallel > e^{-t}$, we have, for a large positive number a,

$$\begin{aligned}
\int_{-\infty}^{0} \parallel \dot{\mathbf{x}}(t) \parallel \ \mathrm{d}t &= \lim_{a \to -\infty} \left(\int_{a}^{0} \parallel \dot{\mathbf{x}}(t) \parallel \ \mathrm{d}t \right) \\
&> \lim_{a \to -\infty} \left(\int_{a}^{0} e^{-t} \, \mathrm{d}t \right) \\
&= \lim_{a \to -\infty} \left\{ - \left(1 - e^{-a} \right) \right\} \\
&= +\infty .
\end{aligned}$$

From above two facts, we see that the improper integral $\displaystyle\int_{-\infty}^{0} \parallel \dot{\mathbf{x}}(t) \parallel \ \mathrm{d}t$ diverges [5]. On the case of $C_n \mid_{-\infty}^{0}$, we have that the length of the curve $C_n \mid_{-\infty}^{0}$ is infinite.

(2) Case of $C_n \mid_{0}^{+\infty}$:
It holds that $\parallel \dot{\mathbf{x}}(t) \parallel > e^{t}$. We have

$$\lim_{b \to +\infty} \int_{0}^{b} e^{t} \, \mathrm{d}t = +\infty .$$

Thus the improper integral $\displaystyle\int_{0}^{+\infty} \parallel \dot{\mathbf{x}}(t) \parallel \ \mathrm{d}t$ diverges [5]. Thus the length of the curve $C_n \mid_{0}^{+\infty}$ is infinite. Therefore, the length of the $C_n \mid_{-\infty}^{+\infty}$ is infinite. This complete the proof.

4 Total first curvature

We consider the arc-length $\varphi(t)$ of the curve $C_n \mid_{-\infty}^{+\infty}$ from the base point $\mathbf{x}(0)$ to the point $\mathbf{x}(t)$. That is, we define

$$\varphi(t) = \int_{0}^{t} \parallel \dot{\mathbf{x}}(t) \parallel \ \mathrm{d}t$$

for any $t \in (-\infty, +\infty)$. We notice that $\varphi(0) = 0$ and $\dfrac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \parallel \dot{\mathbf{x}}(t) \parallel$ for any $t \in (-\infty, +\infty)$. For the curve $C\left\{n : (-\infty, +\infty)\right\}$, its arc-length parameter s is given by

$$s = \varphi(t) = \int_{0}^{t} \parallel \dot{\mathbf{x}}(t) \parallel \ \mathrm{d}t,$$

and s is taken with the sign $+$ if $t > 0$ and with the sign $-$ if $t < 0$. Since $\varphi(t) \to -\infty$ as $t \to -\infty$ and $\varphi(t) \to +\infty$ as $t \to +\infty$, the range of s is $(-\infty, +\infty)$.

Let C be a curve parametrized by arc-length s. Then the *total first curvature* $TC[C]$ of C is defined by

$$TC[C] = \int_I \kappa_1(s)\,\mathrm{d}s,$$

where I denotes the range of arc-length parameter s and κ_1 is the first curvature function of the arc-length parametrized curve C [2].

Hereafter, we consider the total first curvature of the curve $C_n\mid_{-\infty}^{+\infty}$, that is

$$TC[C_n\mid_{-\infty}^{+\infty}] = \int_{-\infty}^{+\infty} \kappa_1(s)\,\mathrm{d}s.$$

This improper integral $\displaystyle\int_{-\infty}^{+\infty} \kappa_1(s)\,\mathrm{d}s$ is rewritten as the form with respect to the original parameter t :

$$\int_{-\infty}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel\, \mathrm{d}t. \tag{†}$$

Let a be a large negative number and b be a large positive number. If both

$$\lim_{b\to+\infty} \int_0^b k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel\, \mathrm{d}t$$

and

$$\lim_{a\to-\infty} \int_a^0 k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel\, \mathrm{d}t$$

exist and are finite, then the improper integral $\displaystyle\int_{-\infty}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel\, \mathrm{d}t$ converges [5], so the curve $C_n\mid_{-\infty}^{+\infty}$ is said to be "a curve *of finite total first curvature*".

As we study whether the total first curvature of $C_n\mid_{-\infty}^{+\infty}$ is convergent or not, we use the form (†) and we rewrite the first curvature function k_1 and $\parallel \dot{\mathbf{x}}(t) \parallel$ as follows:

(1) **In the case:** $n = 2$

$$k_1(t) = \frac{2e^{3t}}{\left(\sqrt{1+e^{4t}}\right)^3}$$

and

$$\parallel \dot{\mathbf{x}}(t) \parallel = e^{-t}\sqrt{1+e^{4t}}$$

for any $t \in (-\infty, +\infty)$.

(2) **In the case:** $n = 2m + 1$

$$
\begin{aligned}
k_1(t) =& e^t \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t} \right)^2 \right. \\
&\left. + 2e^{4t} \left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right) - 2e^{4t} \left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t} \right) \right]^{1/2} \\
&\times \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right) + e^{4t} \right]^{-3/2}
\end{aligned}
$$

and

$$
\| \dot{\mathbf{x}}(t) \| = e^{-t} \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right) + e^{4t} \right]^{1/2}
$$

for any $t \in (-\infty, +\infty)$.

(3) **In the case:** $n = 2m + 2$

$$
\begin{aligned}
k_1(t) =& e^t \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t} \right)^2 \right. \\
& + 2e^{4t} \left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right) - 2e^{4t} \left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t} \right) \\
&\left. + 2 \left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right) + 2 \left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t} \right) + 4e^{4t} \right]^{1/2} \\
&\times \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right) + e^{4t} + 1 \right]^{-3/2}
\end{aligned}
$$

and

$$
\| \dot{\mathbf{x}}(t) \| = e^{-t} \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right) + e^{4t} + 1 \right]^{1/2}
$$

for any $t \in (-\infty, +\infty)$. Here, we notice that $1 + \alpha_k > 0$ for $k = 1, 2, \cdots, m$.

5 Finiteness of total first curvature

We study the finiteness of the total first curvature of the curve $C_n \mid_{-\infty}^{+\infty}$. Let a and b be large positive numbers.

(1) **In the case:** $n = 2$

By Proposition 3, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \mathrm{d}t &= 2\int_{0}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \mathrm{d}t \\ &= 2\int_{0}^{+\infty} \frac{2e^{2t}}{1+e^{4t}}\,\mathrm{d}t \\ &= 2\lim_{b\to+\infty}\left(\int_0^b \frac{2e^{2t}}{1+e^{4t}}\,\mathrm{d}t\right).\end{aligned}$$

If we let $x = f(t) = e^t$ then $\dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t$, $f(0) = 1$ and $f(b) \to +\infty$ $(b \to +\infty)$, so the given integral is rewritten as

$$\lim_{b\to+\infty}\left(\int_0^b \frac{2e^{2t}}{1+e^{4t}}\,\mathrm{d}t\right) = \lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \frac{2x}{1+x^4}\,\mathrm{d}x\right).$$

For any $x \in [1, +\infty)$, we have

$$\begin{aligned}\frac{2}{x^2} - \frac{2x}{1+x^4} &= \frac{2(1+x^4) - 2x^3}{x^2(1+x^4)} \\ &= \frac{2 + 2x^3(x-1)}{x^2(1+x^4)} \\ &> 0.\end{aligned}$$

Thus we have $\dfrac{2x}{1+x^4} < \dfrac{2}{x^2}$ for any $x \in [1, +\infty)$, so it holds

$$\begin{aligned}\lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \frac{2x}{1+x^4}\,\mathrm{d}x\right) &\leq \lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \frac{2}{x^2}\,\mathrm{d}x\right) \\ &= \lim_{f(b)\to+\infty}\left(\frac{-2}{f(b)} + 2\right) \\ &= 2 < +\infty.\end{aligned}$$

Therefore, the improper integreal $\displaystyle\int_{-\infty}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \mathrm{d}t$ converges to a constant number, so the total first curvature of $C_2 \mid_{-\infty}^{+\infty}$ is finite. Thus we have the following:

Proposition 5 *In $\mathbb{E}^2$, the curve $C_2 \mid_{-\infty}^{+\infty}$ is of finite total first curvature.*

Remark We can compute the value of total first curvature of the curve $C_2 \mid_{-\infty}^{+\infty}$. By the discussion in Section 3 (i) $n = 2$, we have

$$\int_{-\infty}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \mathrm{d}t = 2\lim_{b\to+\infty}\left(\int_0^b \frac{2e^{2t}}{1+e^{4t}}\,\mathrm{d}t\right).$$

If we let $x = f(t) = e^{2t}$ then $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2e^{2t}$, $f(0) = 1$ and $f(b) \to +\infty$ $(b \to +\infty)$, so we calculate

$$\begin{aligned}
2 \lim_{b\to+\infty} \left(\int_0^b \frac{2e^{2t}}{1+e^{4t}} \,\mathrm{d}t \right) &= 2 \lim_{f(b)\to+\infty} \left(\int_1^{f(b)} \frac{1}{1+x^2} \,\mathrm{d}x \right) \\
&= 2 \lim_{f(b)\to+\infty} \left([\arctan(x)]_1^{f(b)} \right) \\
&= 2 \left(\frac{\pi}{2} - \frac{\pi}{4} \right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

Therefore, the total first curvature of $C_2 \mid_{-\infty}^{+\infty}$ is equal to $\dfrac{\pi}{2}$.

(2) **In the case:** $n = 2m + 1$

We have

$$\begin{aligned}
\int_{-\infty}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \,\mathrm{d}t =& \int_{-\infty}^{+\infty} K_1(t) \,\mathrm{d}t \\
=& \int_{-\infty}^{0} K_1(t) \,\mathrm{d}t + \int_0^{+\infty} K_1(t) \,\mathrm{d}t \\
=& \lim_{a\to-\infty} \left(\int_a^0 K_1(t) \,\mathrm{d}t \right) + \lim_{b\to+\infty} \left(\int_0^b K_1(t) \,\mathrm{d}t \right),
\end{aligned}$$

where

$$\begin{aligned}
K_1(t) =& k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \\
=& \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t} \right)^2 \right. \\
& \left. + 2e^{4t} \left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t} \right) - 2e^{4t} \left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t} \right) \right]^{1/2} \\
& \times \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k t)} \right) + e^{4t} \right]^{-1}.
\end{aligned}$$

First, we consider the improper integral

$$\int_0^{+\infty} K_1(t) \,\mathrm{d}t = \lim_{b\to+\infty} \left(\int_0^b K_1(t) \,\mathrm{d}t \right).$$

If we let $x = f(t) = e^t$ then $\dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t$, $f(0) = 1$, $f(b) \to +\infty$ $(b \to +\infty)$, so the given integral is rewritten as

$$\lim_{b\to+\infty} \left(\int_0^b K_1(t) \,\mathrm{d}t \right) = \lim_{f(b)\to+\infty} \left(\int_1^{f(b)} \hat{K}_1(x) \,\mathrm{d}x \right),$$

where

$$\hat{K}_1(x) = \left[\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)} \right)^2 \right.$$
$$\left. + 2x^4 \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right) - 2x^4 \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)} \right) \right]^{1/2}$$
$$\times \left[x \left\{ \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right) + x^4 \right\} \right]^{-1}.$$

For any $x \in [1, +\infty)$, we set $\varepsilon = 1 - \alpha_1 > 0$, then we have $2(1 + \alpha_1) = 4 - 2\varepsilon$ so that

$$\sum_{k=1}^{m} x^{2(1+\alpha_k)} \leq \sum_{k=1}^{m} x^{2(1+\alpha_1)} = mx^{2(1+\alpha_1)} = mx^{4-2\varepsilon},$$

where the first equality is satisfied if and only if $x = 1$. Thus we have, for any $x \in (1, +\infty)$,

$$\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right)^2 < m^2 x^{8-4\varepsilon}.$$

For any $x \in [1, +\infty)$, we set $\delta = \dfrac{1}{2}\varepsilon$ and $\dfrac{g(x)}{h(x)} = \dfrac{A}{x^{1+\delta}} - \hat{K}_1(x)$, where $A = \sqrt{m^2 + 4m}$ is a positive constant number. Here

$$g(x) = A^2 \left[\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right) + x^4 \right]^2 - x^{2\delta} \left[\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)} \right)^2 \right.$$
$$\left. + 2x^4 \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right) - 2x^4 \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)} \right) \right]$$

and

$$h(x) = x^{1+\delta} \left[\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right) + x^4 \right] \times \left[A \left\{ \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right) + x^4 \right\} \right.$$
$$+ x^{\delta} \left\{ \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right)^2 - \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)} \right)^2 \right.$$
$$\left. \left. + 2x^4 \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)} \right) - 2x^4 \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)} \right) \right\}^{1/2} \right] > 0.$$

We also have that

$$-x^{2\delta}\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 > -m^2x^{8-4\varepsilon+2\delta} \\ = -m^2x^{8-3\varepsilon} \ > -m^2x^{8-\varepsilon}, \tag{5.1}$$

$$-x^{4+2\delta}\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) > -mx^{4-2\varepsilon+4+2\delta} \\ = -mx^{8-\varepsilon} \tag{5.2}$$

and

$$\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)} > -\sum_{k=1}^{m} x^{2(1+\alpha_k)} \tag{5.3}$$

for any $x \in (1, +\infty)$. From (5.1) (5.2) and (5.3), we obtain

$$\begin{aligned}
g(x) =& A^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 + 2A^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^4 + A^2x^8 \\
& - \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 x^{2\delta} + \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2 x^{2\delta} \\
& - 2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^{4+2\delta} + 2\left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)x^{4+2\delta} \\
>& A^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 + 2A^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^4 + \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2 x^{2\delta} \\
& + A^2x^8 - \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 x^{2\delta} - 4\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^{4+2\delta} \\
>& A^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 + 2A^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^4 + \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2 x^{2\delta} \\
& + A^2x^8 - m^2x^{8-\varepsilon} - 4mx^{8-\varepsilon} \\
=& A^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 + 2A^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^4 + \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2 x^{2\delta} \\
& + x^{8-\varepsilon}\left(A^2x^{\varepsilon} - m^2 - 4m\right).
\end{aligned}$$

Since $A^2 = m^2 + 4m$, then we have $g(x) > 0$, that is, $\dfrac{A}{x^{1+\delta}} > \hat{K}_1(x)$ for any $x \in (1, +\infty)$. Here, $\delta = \dfrac{1-\alpha_1}{2} > 0$. And then we have

$$\begin{aligned}
\lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \frac{\sqrt{m^2+4m}}{x^{1+\delta}}\,\mathrm{d}x\right) =& \sqrt{m^2+4m}\lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \frac{1}{x^{1+\delta}}\,\mathrm{d}x\right)\\
=& \sqrt{m^2+4m}\lim_{f(b)\to+\infty}\left(\left[-\frac{1}{\delta}\frac{1}{x^{\delta}}\right]_1^{f(b)}\right)\\
=& \sqrt{m^2+4m}\lim_{f(b)\to+\infty}\left(-\frac{1}{\delta}\frac{1}{(f(b))^{\delta}} + \frac{1}{\delta}\right)\\
=& \frac{1}{\delta}\sqrt{m^2+4m} \quad < +\infty.
\end{aligned}$$

Thus we have

$$0 < \lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \hat{K}_1(x)\,\mathrm{d}x\right) \le \lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \frac{A}{x^{1+\delta}}\,\mathrm{d}x\right) = \frac{1}{\delta}\sqrt{m^2+4m} < +\infty.$$

Therefore, the improper integral $\displaystyle\int_0^{+\infty} K_1(t)\,\mathrm{d}t = \int_0^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \mathrm{d}t$ converges to a constant number.

Next, we consider the improper integral

$$\int_{-\infty}^{0} K_1(t)\,\mathrm{d}t = \lim_{a\to-\infty}\left(\int_a^0 K_1(t)\,\mathrm{d}t\right).$$

If we let $x = f(t) = e^t$ then $\dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t$, $f(0) = 1$, $f(a) \to 0$ $(a \to -\infty)$, so the given integral is rewritten as

$$\lim_{a\to-\infty}\left(\int_a^0 K_1(t)\,\mathrm{d}t\right) = \lim_{f(a)\to 0}\left(\int_{f(a)}^1 \hat{K}_1(x)\,\mathrm{d}x\right).$$

For any $x \in (0, 1]$, we set $\dfrac{\hat{p}(x)}{\hat{q}(x)} = \hat{K}_1(x) - \dfrac{\hat{A}}{x}$, where $\hat{A} = \sqrt{\dfrac{1-\alpha_1}{2}}$ is a positive constant number. Here

$$\begin{aligned}
\hat{p}(x) = &\left[\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right)^2 - \left(\sum_{k=1}^m \alpha_k x^{2(1+\alpha_k)}\right)^2\right.\\
&\left. + 2x^4\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right) - 2x^4\left(\sum_{k=1}^m \alpha_k x^{2(1+\alpha_k)}\right)\right] - \hat{A}^2\left[\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right) + x^4\right]^2
\end{aligned}$$

and

$$\begin{aligned}\hat{q}(x) =& x\left[\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)+x^4\right]\times\left[\left\{\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2-\left(\sum_{k=1}^{m}\alpha_k x^{2(1+\alpha_k)}\right)^2\right.\right.\\ &\left.+\ 2x^4\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)-2x^4\left(\sum_{k=1}^{m}\alpha_k x^{2(1+\alpha_k)}\right)\right\}^{1/2}\\ &\left.+\ \hat{A}\left\{\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)+x^4\right\}\right]\ >0.\end{aligned}$$

For any $x\in(0,1]$, we have, by the fact: $-1<\alpha_m<\alpha_{m-1}<\cdots<\alpha_2<\alpha_1<1$,

$$\begin{aligned}\hat{p}(x) >& \left[\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2-\left(\sum_{k=1}^{m}\alpha_1 x^{2(1+\alpha_k)}\right)^2\right.\\ &\left.+\ 2x^4\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)-2x^4\left(\sum_{k=1}^{m}\alpha_1 x^{2(1+\alpha_k)}\right)\right]-\hat{A}^2\left[\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)+x^4\right]^2\\ =&(1-(\alpha_1)^2-\hat{A}^2)\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2\\ &+2x^4\left\{\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)-\alpha_1\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)-\hat{A}^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)-\frac{\hat{A}^2}{2}x^4\right\}.\end{aligned}$$

Since $0<2(1+\alpha_k)<4$ and $x\in(0,1]$, we have $x^4\le x^{2(1+\alpha_k)}$ for $k=1,2,\cdots,m$, so that we have

$$x^4\le\frac{1}{m}\sum_{k=1}^{m} x^{2(1+\alpha_k)},$$

where the equalities are satisfied if and only if $x=1$. Thus we have, for any $x\in(0,1)$,

$$\begin{aligned}\hat{p}(x) >&(1-(\alpha_1)^2-\hat{A}^2)\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2\\ &+2x^4\left\{\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)-\alpha_1\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)\right.\\ &\left.-\ \hat{A}^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)-\frac{\hat{A}^2}{2m}\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right\}\\ =&(1-(\alpha_1)^2-\hat{A}^2)\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2\\ &+2x^4\left\{\left(1-\alpha_1-\frac{2m+1}{2m}\hat{A}^2\right)\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right\}.\end{aligned}$$

Since $\hat{A}^2 = \dfrac{1-\alpha_1}{2}$, we have

$$\begin{aligned} 1-(\alpha_1)^2-\hat{A}^2 &= \frac{1}{2}\left\{1+\alpha_1-2(\alpha_1)^2\right\} \\ &= \frac{1}{2}\left\{(1-(\alpha_1)^2)+\alpha_1(1-\alpha_1)\right\} \quad > 0 \end{aligned}$$

and

$$\begin{aligned} 1-\alpha_1-\left(\frac{2m+1}{2m}\right)\hat{A}^2 &= 2\hat{A}^2-\left(\frac{2m+1}{2m}\right)\hat{A}^2 \\ &= \left(\frac{2m-1}{2m}\right)\hat{A}^2 \quad > 0. \end{aligned}$$

Then we have $\hat{p}(x) > 0$ for any $x \in (0,1]$, that is, we have $\hat{K}_1(x) > \dfrac{\hat{A}}{x}$ for $x \in (0,1]$. Therefore, the improper integral $\displaystyle\int_{-\infty}^{0} K_1(t)\,\mathrm{d}t$ diverges. Thus we have the following:

Proposition 6 *In $\mathbb{E}^{2m+1}$, the curve $C_{2m+1}\,|_{-\infty}^{0}$ is of infinite total first curvature and the curve $C_{2m+1}\,|_{0}^{+\infty}$ is of finite total first curvature.*

(3) **In the case:** $n = 2m+2$

We have, by Proposition 3,

$$\begin{aligned} \int_{-\infty}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \;\mathrm{d}t &= 2\int_{0}^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \;\mathrm{d}t \\ &= 2\int_{0}^{+\infty} L_1(t)\,\mathrm{d}t \\ &= 2\lim_{b\to+\infty}\left(\int_0^b L_1(t)\,\mathrm{d}t\right), \end{aligned}$$

where

$$\begin{aligned} L_1(t) = \Bigg[& \left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t}\right)^2 - \left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t}\right)^2 \\ & + 2e^{4t}\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t}\right) - 2e^{4t}\left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t}\right) \\ & + 2\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t}\right) + 2\left(\sum_{k=1}^{m} \alpha_k e^{2(1+\alpha_k)t}\right) + 4e^{4t}\Bigg]^{1/2} \\ & \times \left[\left(\sum_{k=1}^{m} e^{2(1+\alpha_k)t}\right) + e^{4t} + 1\right]^{-1}. \end{aligned}$$

If we let $x = f(t) = e^t$ then $\dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t$, $f(0) = 1$ and $f(b) \to +\infty$ $(b \to +\infty)$, so the given integral is rewritten as

$$\begin{aligned}\int_0^{+\infty} L_1(t)\,\mathrm{d}t &= \lim_{b\to+\infty}\left(\int_0^b L_1(t)\,\mathrm{d}t\right)\\ &= \lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \tilde{L}_1(x)\,\mathrm{d}x\right),\end{aligned}$$

where

$$\begin{aligned}\tilde{L}_1(x) = &\left[\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right)^2 - \left(\sum_{k=1}^m \alpha_k x^{2(1+\alpha_k)}\right)^2\right.\\ &+ 2x^4\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right) - 2x^4\left(\sum_{k=1}^m \alpha_k x^{2(1+\alpha_k)}\right)\\ &\left.+ 2\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right) + 2\left(\sum_{k=1}^m \alpha_k x^{2(1+\alpha_k)}\right) + 4x^4\right]^{1/2}\\ &\times\left[x\left\{\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right) + x^4 + 1\right\}\right]^{-1}.\end{aligned}$$

For any $x \in [1, +\infty)$, we set $\varepsilon = 1 - \alpha_1 > 0$, then we have $2(1+\alpha_1) = 4 - 2\varepsilon$ so that

$$\sum_{k=1}^m x^{2(1+\alpha_k)} \le \sum_{k=1}^m x^{2(1+\alpha_1)} = mx^{2(1+\alpha_1)} = mx^{4-2\varepsilon},$$

where the first equality is satisfied if and only if $x = 1$. Thus we have, for any $x \in (1, +\infty)$,

$$\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right)^2 < m^2 x^{8-4\varepsilon}.$$

For any $x \in [1, +\infty)$, we set $\delta = \dfrac{1}{2}\varepsilon$ and $\dfrac{u(x)}{v(x)} = \dfrac{B}{x^{1+\delta}} - \tilde{L}_1(x)$, where $B = \sqrt{8m^2 + 8m}$ is a positive constant number. Here

$$\begin{aligned}u(x) = &B^2\left[\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right) + x^4 + 1\right]^2 - x^{2\delta}\left[\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right)^2 - \left(\sum_{k=1}^m \alpha_k x^{2(1+\alpha_k)}\right)^2\right.\\ &+ 2x^4\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right) - 2x^4\left(\sum_{k=1}^m \alpha_k x^{2(1+\alpha_k)}\right)\\ &\left.+ 2\left(\sum_{k=1}^m x^{2(1+\alpha_k)}\right) + 2\left(\sum_{k=1}^m \alpha_k x^{2(1+\alpha_k)}\right) + 4x^4\right]\end{aligned}$$

and

$$\begin{aligned}
v(x) =& x^{1+\delta}\left[\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) + x^4 + 1\right] \times \left[B\left\{\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) + x^4 + 1\right\}\right. \\
&+ x^{\delta}\left\{\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 - \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2\right. \\
&+ 2x^4\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) - 2x^4\left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right) \\
&\left.\left.+ 2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) + 2\left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right) + 4x^4\right\}^{1/2}\right] > 0.
\end{aligned}$$

We also have that (5.1), (5.2) and

$$-\sum_{k=1}^{m} x^{2(1+\alpha_k)} < \sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)} < \sum_{k=1}^{m} x^{2(1+\alpha_k)}. \tag{5.4}$$

From (5.1) (5.2) and (5.4), we obtain

$$\begin{aligned}
u(x) =& B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 + B^2x^8 + B^2 \\
&+ 2B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^4 + 2B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) + 2B^2x^4 \\
&- \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 x^{2\delta} + \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2 x^{2\delta} \\
&- 2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^{4+2\delta} + 2\left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)x^{4+2\delta} \\
&- 2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^{2\delta} - 2\left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)x^{2\delta} - 4x^{4+2\delta}
\end{aligned}$$

$$
\begin{aligned}
>&B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 + B^2x^8 + B^2 \\
&+ 2B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^4 + 2B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) + 2B^2x^4 \\
&+ \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2 x^{2\delta} - \left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 x^{2\delta} \\
&- 4\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^{4+2\delta} - 4\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^{2\delta} - 4x^{4+2\delta} \\
>&B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 + B^2x^8 + B^2 \\
&+ 2B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^4 + 2B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) + 2B^2x^4 \\
&+ \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2 x^{2\delta} - m^2x^{8-3\varepsilon} - 4mx^{8-\varepsilon} - 4mx^{4-\varepsilon} - 4x^{4+\varepsilon} \\
>&B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)^2 + B^2 + 2B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right)x^4 \\
&+ 2B^2\left(\sum_{k=1}^{m} x^{2(1+\alpha_k)}\right) + 2B^2x^4 + \left(\sum_{k=1}^{m} \alpha_k x^{2(1+\alpha_k)}\right)^2 x^{2\delta} \\
&+ x^{8-\varepsilon}\left(\frac{B^2}{2}x^{\varepsilon} - m^2 - 4m\right) + x^{4-\varepsilon}\left(\frac{B^2}{2}x^{4+\varepsilon} - 4m - 4x^{2\varepsilon}\right).
\end{aligned}
$$

Since $B^2 = 8m^2 + 8m$, for any $x \in (1, +\infty)$, we have

$$\left(\frac{B^2}{2}x^{\varepsilon} - m^2 - 4m\right) > 0$$

and

$$\left(\frac{B^2}{2}x^{4+\varepsilon} - 4m - 4x^{2\varepsilon}\right) > 0.$$

Then we have $u(x) > 0$, that is, $\dfrac{B}{x^{1+\delta}} > \hat{L}_1(x)$ for any $x \in [1, +\infty)$. Here, $\delta = \dfrac{1-\alpha_1}{2} > 0$. And, similarly in case (2) of Section 5, we have

$$0 < \lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \hat{L}_1(x)\,\mathrm{d}x\right) \le \lim_{f(b)\to+\infty}\left(\int_1^{f(b)} \frac{B}{x^{1+\delta}}\,\mathrm{d}x\right) = \frac{1}{\delta}\sqrt{8m^2+8m} < +\infty.$$

Therefore, the improper integral $2\displaystyle\int_0^{+\infty} L_1(t)\,\mathrm{d}t = 2\int_0^{+\infty} k_1(t) \parallel \dot{\mathbf{x}}(t) \parallel \mathrm{d}t$ converges to a constant number. Thus we have the following:

Proposition 7 *In $\mathbb{E}^{2m+2}$, the curve $C_{2m+2}\,|_{-\infty}^{+\infty}$ is of finite total first curvature.*

Therefore, the Main Theorem is proved by Propositions 4, 5, 6 and 7.

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